

MATA30: Challenge Problem 4.1

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Abstract

In this report I produce a general form for the k^{th} derivative of $e^x \sin x$. As any of its derivatives will be in the form $ae^x \sin x + be^x \cos x$ and the base expression itself can be written in this form if a and b are set to 1 and 0 respectively, I go one step further to generalize the problem and find the general form for the k^{th} derivative of $ae^x \sin x + be^x \cos x$

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1 Introduction

Challenge problem 1 of assignment 4 is:

Find a general formula for $f^{(k)}(x)$ where $f(x) = e^x \sin x$

Although the problem as stated asks only for the k th derivative of

$$e^x \sin x \tag{1}$$

it quickly became apparent that a general formula would be much more useful and elegant if it were applicable to any base expression in the form

$$ae^x \sin x + be^x \cos x \tag{2}$$

So we rephrase the stated problem as:

Find a general formula for $f^{(k)}(x)$ where $f(x) = ae^x \sin x + be^x \cos x$ and a and b are any two given reals

The original question may be re-generated by simply assigning 1 to a and 0 to b .

2 Initial Analysis

To get a look at the general direction this problem was going in, I first manually derived the first few derivatives of $e^x \sin x$. Here is what I saw:

level	derivative
$f^0(x)$	$e^x \sin x$
$f^1(x)$	$e^x \sin x + e^x \cos x$
$f^2(x)$	$2e^x \cos x$
$f^3(x)$	$-2e^x \sin x + 2e^x \cos x$
$f^4(x)$	$-4e^x \sin x$
$f^5(x)$	$-4e^x \sin x + -4e^x \cos x$
$f^6(x)$	$-8e^x \cos x$
$f^7(x)$	$-8e^x \sin x + -8e^x \cos x$
$f^8(x)$	$-16e^x \sin x$

This looks like a pattern is forming, but to get a better look at it I had to include the zero coefficients of my terms. I reworked my table and expanded it to clearly show the coefficients. This time I generalized each derivative, including the initial (zeroth) one, to the form of expression (2), then added columns for the coefficients a_k and b_k

k	$f^k(x)$	a_k	b_k
0	$1e^x \sin x + 0e^x \cos x$	1	0
1	$1e^x \sin x + 1e^x \cos x$	1	1
2	$0e^x \sin x + 2e^x \cos x$	0	2
3	$-2e^x \sin x + 2e^x \cos x$	-2	2
4	$-4e^x \sin x + 0e^x \cos x$	-4	0
5	$-4e^x \sin x + -4e^x \cos x$	-4	-4
6	$0e^x \sin x + -8e^x \cos x$	0	-8
7	$8e^x \sin x + -8e^x \cos x$	8	-8
8	$16e^x \sin x + 0e^x \cos x$	16	0
9	$16e^x \sin x + 16e^x \cos x$	16	16

Here the pattern is clear enough that it looks like we could predict terms. Satisfied that a pattern existed, I set about finding something mathematically provable.

3 The Coefficient Sequence

I predicted that the coefficients of the two terms in the derivatives followed a definable sequence, and set about to define that sequence.

Taking the derivative of $f(x) = a_{k-1}e^x \sin x + b_{k-1}e^x \cos x$ we get:

$$\begin{aligned}
 f'(x) &= a_{k-1}(e^x)(\sin x)' + a_{k-1}(e^x)'(\sin x) \\
 &\quad + b_{k-1}(e^x)(\cos x)' + b_{k-1}(e^x)'(\cos x) \\
 &= a_{k-1}e^x \cos x + a_{k-1}e^x \sin x - b_{k-1}e^x \sin x + b_{k-1}e^x \cos x \\
 &= (a_{k-1} - b_{k-1})e^x \sin x + (a_{k-1} + b_{k-1})e^x \cos x
 \end{aligned}$$

Now we're back to an expression in the form of (2), where the coefficients are $(a_{k-1} - b_{k-1})$ and $(a_{k-1} + b_{k-1})$. Therefore we can state that:

Given coefficients a_{k-1} and b_{k-1} of $f^{(k-1)}(ae^x \sin x + be^x \cos x)$, the coefficients a_k and b_k of the two terms of $f^{(k)}(ae^x \sin x + be^x \cos x)$ are $(a_{k-1} - b_{k-1})$ and $(a_{k-1} + b_{k-1})$.

Now we are on our way to a sequence definition. We can eliminate the b terms from the a_k coefficient like this:

$$\begin{aligned}
 a_k &= a_{k-1} - b_{k-1} \\
 &= a_{k-1} - (a_{k-2} + b_{k-2}) \\
 &= a_{k-1} - (a_{k-2} + (a_{k-3} + b_{k-3})) \\
 &= a_{k-1} - (a_{k-2} + a_{k-3} + a_{k-4} + \dots) \\
 &= a_{k-1} - a_{k-2} - a_{k-3} - a_{k-4} - \dots \\
 &= a_{k-1} - a_{k-2} - a_{k-2} + a_{k-2} - a_{k-3} - a_{k-4} - \dots \\
 &= a_{k-1} - 2a_{k-2} + (a_{k-2} - a_{k-3} - a_{k-4} - \dots)
 \end{aligned}$$

$$\begin{aligned}
&= a_{k-1} - 2a_{k-2} + (a_{k-1}) \\
&= 2a_{k-1} - 2a_{k-2}
\end{aligned}$$

We do something similar for b_k , eliminating the a terms:

$$\begin{aligned}
b_k &= b_{k-1} + a_{k-1} \\
&= b_{k-1} + (-b_{k-2} + a_{k-2}) \\
&= b_{k-1} - b_{k-2} + (-b_{k-3} + a_{k-3}) \\
&= b_{k-1} - b_{k-2} - b_{k-3} - b_{k-4} - \dots \\
&= b_{k-1} - 2b_{k-2} + (b_{k-2} - b_{k-3} - b_{k-4} - \dots) \\
&= b_{k-1} - 2b_{k-2} + b_{k-1} \\
&= 2b_{k-1} - 2b_{k-2}
\end{aligned}$$

I was pleasantly surprised to see that the sequence definition was the same for a and b . In general terms, this is

$$n_k = 2n_{k-1} - 2n_{k-2} \tag{3}$$

4 The Coefficient Formula

Early on I noticed that the coefficients of the $e^x \cos c$ terms in the derivatives seemed to be following an exponentially growing sine function. A little examination led to the observation that for $f^k(e^x \sin x)$, the coefficient of the second term could be described by

$$\sqrt{2}^k \sin \frac{\pi k}{4}$$

Similarly, the coefficient of the first term could be described by

$$\sqrt{2}^k \cos \frac{\pi k}{4}$$

The trouble was, I had no link between that observation and the sequence I had come up with. In addition, those two expressions only held true for $a = 1; b = 0$. I had to develop a better solution.

I wrote a short Python programA to generate sequences of coefficients for different values of a and b . This way, I could create mutations of the sine function I had come up with and test them. After many wrong turns and dead-ends, I looked at the original formula I came up with and wondered what happened if I took the number in the sequence I had come up with and normalized them by divided by $\sqrt{2}^k$ — perhaps the one constant element in all the formulae I was trying. What I found was that when this expression was factored out the coefficients followed a sine curve with some interesting values.

k	a_k	$\frac{a_k}{\sqrt{2^k}}$	b_k	$\frac{b_k}{\sqrt{2^k}}$
0	5.000	5.000	2.000	2.000
1	3.000	2.121	7.000	4.950
2	-4.000	-2.000	10.000	5.000
3	-14.000	-4.950	6.000	2.121
4	-20.000	-5.000	-8.000	-2.000
5	-12.000	-2.121	-28.000	-4.950
6	16.000	2.000	-40.000	-5.000
7	56.000	4.950	-24.000	-2.121
8	80.000	5.000	32.000	2.000

I noticed that the normalized values of a_k followed the pattern:

$$\left(a, \frac{\sqrt{2}}{2} (a - b), -b, \frac{\sqrt{2}}{2} (-a - b), -a, \frac{\sqrt{2}}{2} (-a + b), b, \frac{\sqrt{2}}{2} (+a + b) \right)$$

I soon realized that this was because the secret sine functions I was looking for were:

$$a_k = \sqrt{2^k} \left(a \cos \frac{\pi k}{4} - b \sin \frac{\pi k}{4} \right) \quad (4)$$

$$b_k = \sqrt{2^k} \left(a \sin \frac{\pi k}{4} + b \cos \frac{\pi k}{4} \right) \quad (5)$$

So finally I had an expressible proof to try and solve.

5 Solution

With a sequence that has been shown to describe the coefficients of the terms in the derivatives I was examining, and a pair of formulae that I had empirically observed seems to describe the same sequence, I needed only prove that any value of function (4) at k could be described by the sequence definition in terms of the value of function (4) at $k - 1$ and $k - 2$, and do the same thing with function (5).

5.1 Coefficients of sine term

Show that for all $a, b \in R; k \in I$,

$$\begin{aligned} \sqrt{2}^k \left(a \cos \frac{k\pi}{4} - b \sin \frac{k\pi}{4} \right) &= 2\sqrt{2}^{k-1} \left(a \cos \frac{(k-1)\pi}{4} - b \sin \frac{(k-1)\pi}{4} \right) \\ &\quad - 2\sqrt{2}^{k-2} \left(a \cos \frac{(k-2)\pi}{4} - b \sin \frac{(k-2)\pi}{4} \right) \end{aligned}$$

$$\begin{aligned} &2\sqrt{2}^{k-1} \left(a \cos \left(\frac{(k-1)\pi}{4} \right) - b \sin \left(\frac{(k-1)\pi}{4} \right) \right) \\ &- 2\sqrt{2}^{k-2} \left(a \cos \left(\frac{(k-2)\pi}{4} \right) - b \sin \left(\frac{(k-2)\pi}{4} \right) \right) \\ = &\sqrt{2}\sqrt{2}^k \left(a \cos \left(\frac{(k-1)\pi}{4} \right) - b \sin \left(\frac{(k-1)\pi}{4} \right) \right) \\ &- \sqrt{2}^k \left(a \cos \left(\frac{(k-2)\pi}{4} \right) - b \sin \left(\frac{(k-2)\pi}{4} \right) \right) \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \cos \left(\frac{(k-1)\pi}{4} \right) - b \sin \left(\frac{(k-1)\pi}{4} \right) \right) \\ - \left(a \cos \left(\frac{(k-2)\pi}{4} \right) - b \sin \left(\frac{(k-2)\pi}{4} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \cos \left(\frac{k\pi}{4} - \frac{\pi}{4} \right) - b \sin \left(\frac{k\pi}{4} - \frac{\pi}{4} \right) \right) \\ - \left(a \cos \left(\frac{k\pi}{4} - \frac{\pi}{2} \right) - b \sin \left(\frac{k\pi}{4} - \frac{\pi}{2} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \left[\cos \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{4} \right) - \sin \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{4} \right) \right] \right) \\ - \left(a \left[\cos \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{2} \right) - \sin \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{2} \right) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \left[\cos \left(\frac{k\pi}{4} \right) \left(\frac{\sqrt{2}}{2} \right) - \sin \left(\frac{k\pi}{4} \right) \left(-\frac{\sqrt{2}}{2} \right) \right] \right) \\ - \left(a \left[\cos \left(\frac{k\pi}{4} \right) (0) - \sin \left(\frac{k\pi}{4} \right) (-1) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \left(a \left[\cos \left(\frac{k\pi}{4} \right) + \sin \left(\frac{k\pi}{4} \right) \right] - b \left[\sin \left(\frac{k\pi}{4} \right) - \cos \left(\frac{k\pi}{4} \right) \right] \right) \\ - \left(a \left[+ \sin \left(\frac{k\pi}{4} \right) \right] - b \left[- \cos \left(\frac{k\pi}{4} \right) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \left(a \cos \left(\frac{k\pi}{4} \right) + a \sin \left(\frac{k\pi}{4} \right) - b \sin \left(\frac{k\pi}{4} \right) + b \cos \left(\frac{k\pi}{4} \right) \right) \\ - \left(a \sin \left(\frac{k\pi}{4} \right) + b \cos \left(\frac{k\pi}{4} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left(a \cos \left(\frac{k\pi}{4} \right) - b \sin \left(\frac{k\pi}{4} \right) \right) \end{aligned}$$

5.2 Coefficients of cosine term

Show that for all $a, b \in R; k \in I$,

$$\begin{aligned} \sqrt{2}^k \left(a \sin \frac{k\pi}{4} - b \cos \frac{k\pi}{4} \right) &= 2\sqrt{2}^{k-1} \left(a \sin \frac{(k-1)\pi}{4} - b \cos \frac{(k-1)\pi}{4} \right) \\ &\quad - 2\sqrt{2}^{k-2} \left(a \sin \frac{(k-2)\pi}{4} - b \cos \frac{(k-2)\pi}{4} \right) \end{aligned}$$

$$\begin{aligned} &2\sqrt{2}^{k-1} \left(a \sin \left(\frac{(k-1)\pi}{4} \right) - b \cos \left(\frac{(k-1)\pi}{4} \right) \right) \\ &- 2\sqrt{2}^{k-2} \left(a \sin \left(\frac{(k-2)\pi}{4} \right) - b \cos \left(\frac{(k-2)\pi}{4} \right) \right) \\ = &\sqrt{2}\sqrt{2}^k \left(a \sin \left(\frac{(k-1)\pi}{4} \right) - b \cos \left(\frac{(k-1)\pi}{4} \right) \right) \\ &- \sqrt{2}^k \left(a \sin \left(\frac{(k-2)\pi}{4} \right) - b \cos \left(\frac{(k-2)\pi}{4} \right) \right) \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \sin \left(\frac{(k-1)\pi}{4} \right) - b \cos \left(\frac{(k-1)\pi}{4} \right) \right) \\ - \left(a \sin \left(\frac{(k-2)\pi}{4} \right) - b \cos \left(\frac{(k-2)\pi}{4} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \sin \left(\frac{k\pi}{4} - \frac{\pi}{4} \right) - b \cos \left(\frac{k\pi}{4} - \frac{\pi}{4} \right) \right) \\ - \left(a \sin \left(\frac{k\pi}{4} - \frac{\pi}{2} \right) - b \cos \left(\frac{k\pi}{4} - \frac{\pi}{2} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \left[\sin \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{4} \right) + \cos \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{4} \right) \right] \right. \\ \left. - b \left[\cos \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{4} \right) - \sin \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{4} \right) \right] \right) \\ - \left(a \left[\sin \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{2} \right) + \cos \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{2} \right) \right] \right. \\ \left. - b \left[\cos \left(\frac{k\pi}{4} \right) \cos \left(-\frac{\pi}{2} \right) - \sin \left(\frac{k\pi}{4} \right) \sin \left(-\frac{\pi}{2} \right) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \sqrt{2} \left(a \left[\sin \left(\frac{k\pi}{4} \right) \left(\frac{\sqrt{2}}{2} \right) + \cos \left(\frac{k\pi}{4} \right) \left(-\frac{\sqrt{2}}{2} \right) \right] \right. \\ \left. - b \left[\cos \left(\frac{k\pi}{4} \right) \left(\frac{\sqrt{2}}{2} \right) - \sin \left(\frac{k\pi}{4} \right) \left(-\frac{\sqrt{2}}{2} \right) \right] \right) \\ - \left(a \left[\sin \left(\frac{k\pi}{4} \right) (0) - \cos \left(\frac{k\pi}{4} \right) (-1) \right] \right. \\ \left. - b \left[\cos \left(\frac{k\pi}{4} \right) (0) + \sin \left(\frac{k\pi}{4} \right) (-1) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \left(a \left[\sin \left(\frac{k\pi}{4} \right) - \cos \left(\frac{k\pi}{4} \right) \right] - b \left[\cos \left(\frac{k\pi}{4} \right) + \sin \left(\frac{k\pi}{4} \right) \right] \right) \\ - \left(a \left[+ \cos \left(\frac{k\pi}{4} \right) \right] - b \left[- \sin \left(\frac{k\pi}{4} \right) \right] \right) \end{array} \right] \\ = &\sqrt{2}^k \left[\begin{array}{l} \left(a \sin \left(\frac{k\pi}{4} \right) - a \cos \left(\frac{k\pi}{4} \right) - b \cos \left(\frac{k\pi}{4} \right) - b \sin \left(\frac{k\pi}{4} \right) \right) \\ - \left(a \cos \left(\frac{k\pi}{4} \right) + b \sin \left(\frac{k\pi}{4} \right) \right) \end{array} \right] \\ = &\sqrt{2}^k \left(a \sin \left(\frac{k\pi}{4} \right) - b \cos \left(\frac{k\pi}{4} \right) \right) \end{aligned}$$

A challenge_4_1.py

```
#!/usr/bin/env python

import sys, math, string
from math import sqrt, pi, sin, cos

root2 = sqrt(2)

def progression(a0=0.0,b0=0.0,x=1,y=1):
    a0 = float(a0)
    b0 = float(b0)
    sys.stdout.write("\n")
    for a in range(0,y):
        ak = {}
        bk = {}
        for b in range(0,x):
            ak[b] = a0 + float(a)
            bk[b] = b0 + float(b)
            sys.stdout.write("k".rjust(4))
            sys.stdout.write("a_k".rjust(7))
            sys.stdout.write("norm a".rjust(7))
            sys.stdout.write("func a".rjust(7))
            sys.stdout.write("b_k".rjust(7))
            sys.stdout.write("norm b".rjust(7))
            sys.stdout.write("func b".rjust(7))
        sys.stdout.write( "\n")
        for k in range(0, 9):
            for b in range(0,x):
                sys.stdout.write(str("%d:" % k).rjust(4))
                sys.stdout.write(str("%0.3f" % ak[b]).rjust(7))
                sys.stdout.write(str("%0.3f" % \
                    norm(ak[b],k)).rjust(7))
                sys.stdout.write(str("%0.3f" % \
                    sinFormula(k,a0 + float(a),b0 + float(b)) \
                    ).rjust(7))

                sys.stdout.write(str("%0.3f" % bk[b]).rjust(7))
                sys.stdout.write(str("%0.3f" % \
                    norm(bk[b],k)).rjust(7))
                sys.stdout.write(str("%0.3f" % \
                    cosFormula(k,a0 + float(a),b0 + float(b)) \
                    ).rjust(7))
            ak[b] = ak[b] - bk[b]
            bk[b] = ak[b] + 2*bk[b]
        sys.stdout.write( "\n")
```

```

    sys.stdout.write( "\n")

def sinFormula(k,a,b):
    k = float(k)
    a = float(a)
    b = float(b)
    return pow(root2,k)*(a*cos(k*pi/4)-b*sin(k*pi/4))
def cosFormula(k,a,b):
    k = float(k)
    a = float(a)
    b = float(b)
    return pow(root2,k)*(b*cos(k*pi/4)+a*sin(k*pi/4))

def norm(n,k):
    return float(n) / pow(2,float(k)/2)

if (__name__ == "__main__"):
    import re
    count = 15
    if not len(sys.argv)>1:
        squareProgression(3,4)
    for arg in sys.argv[1:]:
        mo = re.match("^(?P<x>\d+),(?P<y>\d+),)"+"
            "(?P<a>-?\d+(\.\d+)?) ,(?P<b>-?\d+(\.\d+)?)$", arg)
        if not mo:
            print "arg " + arg + " generated no match"
        else:
            x = 1
            y = 1
            a = float(mo.group("a"))
            b = float(mo.group("b"))
            if mo.group("x"):
                x = int(mo.group("x"))
                y = int(mo.group("y"))
            progression(a, b, x, y)

```